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# Assessing Risk of a Serious Failure Mode Based on Limited Field Data

## Abstract

Nowadays, many consumer products are designed and manufactured so that the probability of failure during the technological life of the product is small. Most product units in the field retire before they fail. Even though the number of failures of such products is small, there is still a need to model and predict field failures for purposes of risk assessment in applications that involve safety. Challenges in modeling and predictions of failures arise because the retirement times are often unknown, few failures have been reported, and there are delays in field failure reporting. Motivated by an application to assess the risk of failure for a particular product, we develop a statistical prediction procedure that considers the impact of product retirements and reporting delays. Based on the developed method, we provide the point predictions for cumulative number of reported failures over a future time period and corresponding prediction intervals to quantify uncertainty. We also conduct sensitivity analysis to assess the effects of different assumptions on failure-time and retirement distributions.

## Keywords

calibration, discrete Fourier transform, failure reporting delay, Poisson-binomial distribution, prediction interval, Weibull

## Disciplines

Statistics and Probability

## Comments

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# Assessing Risk of a Serious Failure Mode Based on Limited Field Data

Zhibing Xu, Yili Hong, and William Q. Meeker

## Abstract

Nowadays, many consumer products are designed and manufactured so that the probability of failure during the technological life of the product is small. Most product units in the field retire before they fail. Even though the number of failures of such products is small, there is still a need to model and predict field failures for purposes of risk assessment in applications that involve safety. Challenges in modeling and predictions of failures arise because the retirement times are often unknown, few failures have been reported, and there are delays in field failure reporting. Motivated by an application to assess the risk of failure for a particular product, we develop a statistical prediction procedure that considers the impact of product retirements and reporting delays. Based on the developed method, we provide the point predictions for cumulative number of reported failures over a future time period and corresponding prediction intervals to quantify uncertainty. We also conduct sensitivity analysis to assess the effects of different assumptions on failure-time and retirement distributions.

## Index Terms

Calibration, Discrete Fourier transform, Failure reporting delay, Poisson-binomial distribution, Prediction interval, Weibull.

## ACRONYMS

cdf	cumulative distribution function
DFD	data freeze date
ML	maximum likelihood
pdf	probability density function
PI	prediction interval

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## NOTATION

$N(s)$	cumulative number of reported failures at time $s$
$R$	retirement time
$T$	failure time
$\Delta$	reporting delay time
$f_T(t)$	pdf of the failure-time distribution
$F_T(t)$	cdf of the failure-time distribution
$F_R(r)$	cdf of the retirement-time distribution
$\pi_j$	probability of a reported failure in batch $j$ before DFD
$\rho_j(s)$	probability of a unit in batch $j$ being reported as a failure in $(0, s]$
$\Phi_{\text{sev}}$	standard smallest extreme value cdf
$\phi_{\text{sev}}$	standard smallest extreme value pdf
$\Phi_{\text{nor}}$	standard normal cdf
$\phi_{\text{nor}}$	standard normal pdf
$E(R)$	expected value (mean) of the retirement-time distribution
$\text{Var}(R)$	variance of the retirement-time distribution
$\text{SD}(R)$	standard deviation of the retirement-time distribution
$\beta_T$	Weibull shape parameter of the failure-time distribution
$\eta_T$	Weibull scale parameter of the failure-time distribution
$\mu_T$	location parameter of the distribution of $\log(T)$
$\sigma_T$	scale parameter of the distribution of $\log(T)$
$\xi_j$	probability of a unit in batch $j$ not being reported as failing before DFD
$\mathcal{A}_j$	age of a unit in batch $j$ at the DFD
$w_j$	number of units not being reported as failures before DFD in batch $j$
$\hat{\theta}$	ML estimator of $\theta$
$\mathcal{L}(\theta   \text{DATA})$	likelihood function for the unknown parameter vector $\theta$

## I. INTRODUCTION

### A. Motivation

Product reliability is important to both the manufacturers and consumers. Many products nowadays have high reliability with only a small fraction failing (e.g., 1% or less). There are, however, some failure modes that can lead to risk of loss of property or life. Examples include material anomalies in rotating components in aircraft engines that lead to premature

cracking and fracture, failure of electrical insulation in home appliances giving rise to risk of fire or electrical shock, failure of electrical connections in defibrillators, explosion of a laptop battery, and so on.

Although the particular technical details and nature of the available data and other information will differ from application to application, there is a common scenario that we have seen in numerous different applications. At some point in time (which may range from months to years) after product introduction, a few failures have been reported. Often the particular failure mode is one that had not been anticipated. Sometimes the problem was caused by just a single batch of raw material or an unreported and untested change in a component or material made by a vendor. Generally management (or in some cases government agencies) will want engineers to determine whether there is a serious problem and will often ask for a formal risk assessment. This then leads to the asking of some or all of the following questions.

- 1) Were the reported failures anomalies (e.g., cause by extreme product abuse or a few defective units that got shipped) or is the problem more widespread? Usually, it is the latter, but wishful thinking will cause some to believe the former.
- 2) Is there a small proportion of defective units failing rapidly or will all units (that remain in service) eventually fail prematurely?
- 3) What is the risk (e.g., potential cost, both tangible and intangible) of future failures from this product?
- 4) Should there be a product recall?
- 5) How can we fix the problem so that future production will not have the failure mode of concern?

This paper focuses on statistical methods for answering question 3.

In some applications, the risk of failure is lessened because of product retirement, before product failure occurs. Retirements are often a result of product performance degradation or technical obsolescence. For example, cell phones and laptop computers are typically retired after two or three years of use. Ironically, for some products, the risk of a serious failure is sometimes lessened by the occurrence of an innocuous failure mode. For example, an implanted defibrillator that has a broken electrical connection would be removed from service if its rechargeable battery fails before the unit is called upon to be used. In such applications, possible retirement/innocuous failure events should be part of the risk assessment. Another complicating feature of some field data is the delayed reporting of failures.

Motivated by several different but similar applications, we develop a statistical procedure

to predict the field failures of products, considering the impact of product retirement and reporting delays. Based on the developed method, we provide point predictions for cumulative number of reported failures at a future point in time and the corresponding prediction interval (PI) to quantify uncertainty.

### B. Related work

There is a large amount of literature describing statistical prediction and some of this previous work has focused on the prediction of the number of failures in a future time period and the construction of a corresponding PI. Nelson [1], and Meeker and Escobar [2] introduced general methods to obtain PIs for reliability applications. Engehardt and Bain [3] provided an exact PI for the number of failures in a repairable system based on maximum likelihood (ML) estimation. Mee and Kushary [4] gave simulation-based methods for computing PIs for selected order statistics from future samples from a Weibull distribution. Nelson [5] and Nordman and Meeker [6] proposed PI procedures based on a Weibull distribution with a known shape parameter. Geisser [7] and Tian, Tang, and Yu [8] described Bayesian approaches to obtain a PI. De Menezes, Vivanco, and Sampaio [9] used subsampling to obtain the PI for the number of failures in a future time interval. For censored failure-time data, Escobar and Meeker [10], and Hong, Meeker, and McCalley [11] described methods to obtain PIs for a future number of failures. Lawless and Fredette [12] proposed an effective and easy-to-use procedure to construct frequentist PIs. Few published works, however, have considered prediction in the presence of the unknown retirement times and reporting delays. In one exception, Zhao, Steffey, and Loud [13] compared the difference of predictions between the models that account for retirement and that do not account for retirement, based on a specific retirement rate assumption. In this paper, we propose a general statistical procedure to predict the future number of field failures in the presence of retirement and reporting delays and we develop a PI procedure to quantify the uncertainties in prediction.

### C. Overview

Our approach to the field-failure prediction uses the following steps:

- *Failure-time modeling:* We first construct a failure-time model based on assumptions for the retirement-time distribution and reporting delays. Then, we estimate the parameters of the failure-time distribution using ML.
- *Derivation of the probability of future failures:* Based on the failure-time and retirement-time distributions, as well as the ML estimates from Step 1, the probability of a reported

failure in a future time interval can be estimated, providing the basis of prediction of the cumulative number of reported failures in a specified future time period.

- *Prediction:* Based on the probability of a reported failure and the number of units that are at risk, one can obtain a point prediction for the cumulative number of reported failures by a specified future point in time and a corresponding PI.
- *Prediction Interval:* We use a method based on the concept of a predictive distribution and bootstrap calibration to construct PIs for the future number of failures.
- *Sensitivity analysis:* The predictions are based on uncertain assumptions about the failure-time and the retirement-time distributions. Thus, it is prudent to assess the effect of deviations from these assumptions.

The rest of the paper is organized as follows. Section II introduces the failure-time distribution, the retirement-time distribution, and the reporting delay distribution. Section III develops an ML procedure to estimate the unknown failure-time distribution parameters. Section IV shows in detail how to use the failure-time distribution and estimated parameters to predict the number of reported future failures and how to compute a corresponding PI. In Section V, sensitivity analysis is used to compare the prediction results with different parameters and distributions. Section VI gives some concluding remarks and describes possible areas for future research.

## II. DATA AND FAILURE-TIME MODEL

### A. The Data

This paper uses a dataset from a product that is used at home that we call product B. To protect proprietary and sensitive information, we have disguised the data by changing the time scale and using a randomly chosen subset of the original dataset. Although our methods were motivated by this specific application, the developed method is general and can be applied to other situations with unknown retirement times and reporting delays.

The company manufactured 14 batches of product B over time and there were 120,921 units in total. The units were put into service at different times between January 1996 and 1997 (staggered entry). We define the first installation time (i.e., January 1996) to be time 0. Then the installation times for the 14 different batches were 0, 2, 3, 4, 5, 6, 7, 10, 12, 13, 14, 15, 16 and 17 months, respectively. After 55 months (a little less than five years) later, a potentially dangerous failure mode was reported. Subsequently 32 additional failures were reported by the data freeze date (DFD), which was at 118 months (about 9.8 years) after the first units were introduced into service. Figure 1 illustrates the staggered entry pattern. The

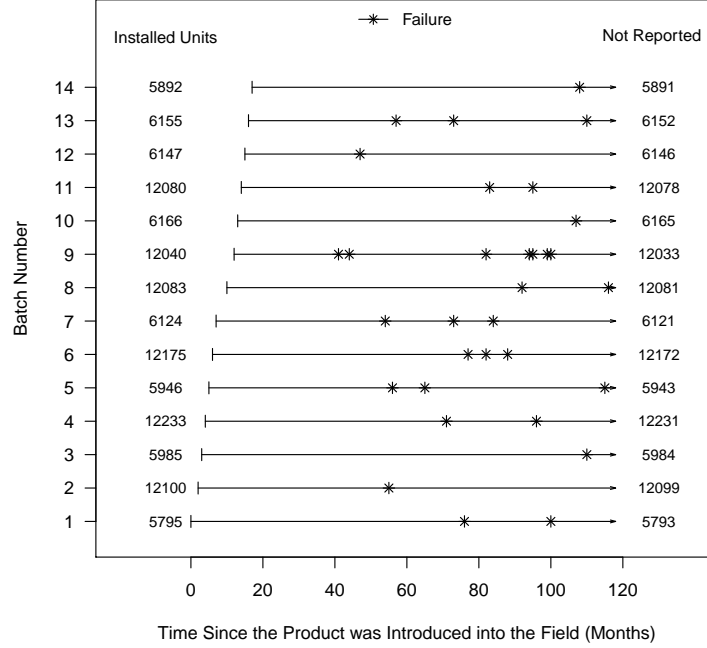


Figure 1. Event plot for the dataset with “\*” indicating the failure times for the reported failures.

figure shows, for each batch, the number of units that had been installed, the time where failures were reported, and the number of units that had not being reported as failing by the DFD.

Table I shows the 32 reported failure times (months in service before failure). The failure times are denoted by  $t_i, i = 1, \dots, r$  where  $r$  is the total number of reported failures ( $r = 32$  here). The failure times were recorded to the nearest month. For example, failure time 91 indicates that a unit failed between 90.5 and 91.5 months after its installation. Table I also lists the age that the failed unit would have been at the DFD if it had not failed; these times are denoted by  $\mathcal{A}_i, i = 1, \dots, r$ .

Table II shows the number of units installed, the number of failures reported, the number of units not being reported by the DFD, and the ages of units at the DFD (denoted by  $\mathcal{A}_j$ ) for the 14 batches of product B. The number of units installed, denoted by  $n_j, j = 1, \dots, J$  where  $J$  is the total number of batches ( $J = 14$  here), ranges between 5,795 and 12,233. The number of units that were not reported is denoted by  $w_j$ . The number of reported failures from each batch and the overall fraction failing is small.



Table I  
FAILURE TIME  $t_i$  AND CORRESPONDING AGE OF THE UNIT'S BATCH AT THE DFD  $\mathcal{A}_i$ .

$t_i$	$\mathcal{A}_i$	$t_i$	$\mathcal{A}_i$	$t_i$	$\mathcal{A}_i$	$t_i$	$\mathcal{A}_i$
91	101	88	106	106	108	60	113
41	102	82	106	47	111	51	113
94	102	32	106	77	111	92	114
57	102	29	106	66	111	67	114
32	103	83	106	76	112	107	115
69	104	87	106	82	112	53	116
81	104	70	106	71	112	100	118
94	105	82	108	110	113	76	118

Table II  
INSTALLED QUANTITIES, NUMBER OF FAILURES REPORTED, NUMBER NOT REPORTED, AND THE AGE OF THE BATCH AT THE DFD.

Batch $j$	Installed $n_j$	Reported $r_j$	Not Reported $w_j = n_j - r_j$	Batch Age at DFD $\mathcal{A}_j$
1	5795	2	5793	118
2	12100	1	12099	116
3	5985	1	5984	115
4	12233	2	12231	114
5	5946	3	5943	113
6	12175	3	12172	112
7	6124	3	6121	111
8	12083	2	12081	108
9	12040	7	12033	106
10	6166	1	6165	105
11	12080	2	12078	104
12	6147	1	6146	103
13	6155	3	6152	102
14	5892	1	5891	101

### B. Model for Time to Failure

Let  $T$  denote the product failure time. We use the log-location-scale family of distributions to model the distribution of  $T$ . Among those members in the log-location-scale family, the Weibull and lognormal distributions are the two most commonly-used distributions for describing failure times. In particular, the cumulative distribution function (cdf) and probability

density function (pdf) of the Weibull distribution can be expressed as

$$F_T(t; \mu_T, \sigma_T) = \Phi_{\text{sev}} \left[ \frac{\log(t) - \mu_T}{\sigma_T} \right] \quad \text{and} \quad f_T(t; \mu_T, \sigma_T) = \frac{1}{\sigma_T t} \phi_{\text{sev}} \left[ \frac{\log(t) - \mu_T}{\sigma_T} \right] \quad t > 0,$$

where  $\Phi_{\text{sev}}(w) = 1 - \exp[-\exp(w)]$  and  $\phi_{\text{sev}}(w) = \exp[w - \exp(w)]$  are the standard smallest extreme value cdf and pdf, respectively. Here,  $\mu_T$  is the location parameter, and  $\sigma_T$  is the scale parameter of the distribution of  $\log(T)$ .

We use the Weibull distribution to describe the distribution of  $T$ . In our sensitivity analysis, the lognormal distribution is considered as an alternative to describe the failure-time distribution. By replacing  $\Phi_{\text{sev}}$  and  $\phi_{\text{sev}}$  with  $\Phi_{\text{nor}}$  and  $\phi_{\text{nor}}$ , the standard normal cdf and pdf, the cdf and pdf of the lognormal distribution are

$$F_T(t; \mu_T, \sigma_T) = \Phi_{\text{nor}} \left[ \frac{\log(t) - \mu_T}{\sigma_T} \right] \quad \text{and} \quad f_T(t; \mu_T, \sigma_T) = \frac{1}{\sigma_T t} \phi_{\text{nor}} \left[ \frac{\log(t) - \mu_T}{\sigma_T} \right] \quad t > 0,$$

respectively.

The cdf and pdf of the Weibull distribution can also be re-expressed as:

$$F_T(t; \eta_T, \beta_T) = 1 - \exp \left[ - \left( \frac{t}{\eta_T} \right)^{\beta_T} \right] \quad \text{and} \quad f_T(t; \eta_T, \beta_T) = \frac{\beta_T}{\eta_T} \left( \frac{t}{\eta_T} \right)^{\beta_T - 1} \exp \left[ - \left( \frac{t}{\eta_T} \right)^{\beta_T} \right],$$

where  $t > 0$ ,  $\eta_T = \exp(\mu_T)$  is the Weibull scale parameter (also the approximate 0.63 quantile), and  $\beta_T = 1/\sigma_T$  is the Weibull shape parameter. The value of  $\beta_T$  indicates the shape of the hazard function, which is given by

$$h(t; \beta_T, \eta_T) = \frac{f(t; \beta_T, \eta_T)}{1 - F(t; \beta_T, \eta_T)} = \frac{\beta_T}{\eta_T} \left( \frac{t}{\eta_T} \right)^{\beta_T - 1}.$$

In particular,  $\beta_T > 1$  indicates an increasing hazard function;  $\beta_T = 1$  indicates a constant hazard function; and  $\beta_T < 1$  indicates a decreasing hazard function.

### C. Product Retirement Distribution

Product retirement occurs when a unit is removed from service before it fails. Let  $R$  be the time of retirement. To avoid prediction bias, it is important to incorporate the retirement information into the failure-time model. However, there is no tracking of an individual product's retirement time. Other information on the retirement distribution at population level has to be used.

Based on information available from a previously conducted marketing survey, it was believed that the retirement distribution could be described by a Weibull distribution

$$F_R(r) = \Pr(R \leq r) = 1 - \exp \left[ - \left( \frac{r}{\eta_R} \right)^{\beta_R} \right], \quad r > 0 \quad (1)$$

Table III  
PROBABILITY MASS FUNCTION FOR THE DISTRIBUTION OF REPORTING DELAYS.

$\delta$ (months)	0	1	2	3 - 5	6 - 9	10 - 15	$\geq 16$
$\Pr(\Delta = \delta)$	0.62	0.31	0.04	0.004	0.003	0.001	0

with a mean  $E(R) = 98$  months (approximately 8.2 years) and a shape parameter  $\beta_R$  between 1.5 and 2. The value of shape parameter  $\beta_R > 1$  indicates that the retirement hazard function is an increasing function of product age. The mean of the Weibull distribution is  $E(R) = \eta_R \Gamma(1 + 1/\beta_R)$ , implying that the Weibull “characteristic life” parameter  $\eta_R$  is between 108.6 and 110.6 months. Due to the nature of the serious failure mechanism (because it has no symptoms before it occurs), it can reasonably be assumed to be independent of the time of retirement.

#### D. Failure Reporting Delay

In this application there were known delays in the reporting of failures. Because these delays were potentially important to the estimation of the failure-time distribution, there was need to consider them in modeling and prediction. We denote the length of the delay by  $\Delta$ , which is assumed to be independent of the failure time. The reporting time is equal to  $T + \Delta$ , where  $T$  is product’s failure time.

Based on available records, no reporting delays had been longer than 15 months. Thus, the probability of a reporting delay greater than or equal to 16 months is equal to zero, and all of delay times are between 0 month to 15 months. Based on historical information, the distribution of delays is approximated by a discrete distribution given in Table III. A particular delay time is denoted by  $\delta$  and the corresponding probability is denoted by  $\Pr(\Delta = \delta)$ . Table III indicates that around 62% of failures would be reported to the company without any delay. Note that  $\sum_{\delta} \Pr(\Delta = \delta) = 1$ . Because the failure process and reporting process are not related, it is reasonable to assume that the reporting delay  $\Delta$  is independent of failure time  $T$ .

### III. MAXIMUM LIKELIHOOD ESTIMATION

#### A. Construction of the Likelihood Function

Let  $t_i$  denote the realized failure time of unit  $i$ , which is the amount of time between when the unit was installed and when it failed. If a unit failed and the failure was reported

before the DFD, the failure that occurred at time  $t_i$  was recorded. Because the failure times were recorded to the nearest month, the actual failure time for observation  $i$  is in the interval  $(t_i - .5, t_i + .5)$ . The probability of a failure before retirement with failure time between  $t_i - .5$  and  $t_i + .5$  is

$$\Pr[(T \leq R) \cap (t_i - .5 < T \leq t_i + .5)] = \int_{t_i - .5}^{t_i + .5} f_T(t)[1 - F_R(t)]dt \quad (2)$$

where the factor  $1 - F_R(t)$  represents the probability that the unit retires after time  $t$ . One can consider the failure time  $T$  and retirement time  $R$  as in a competing-risks model (e.g., Crowder [14]). Figure 2 illustrates the computing of the likelihood contribution in (2) in which the shaded area shows the likelihood contribution.

To account for reporting delay, (2) needs to be modified. Here, the reporting delay is incorporated into the model by conditioning on the observed value of  $\Delta$ . In particular, the probability of actually failing in the interval  $(t_i - .5, t_i + .5)$  and having the failure reported before the DFD is

$$\begin{aligned} \pi_i &= \Pr[(T \leq R) \cap (t_i - .5 < T \leq t_i + .5) \cap \text{Reported}] \\ &= \sum_{\delta} \Pr(\Delta_i = \delta) \Pr[(T \leq R) \cap (t_i - .5 < T \leq t_i + .5) \cap \text{Reported} | \Delta_i = \delta] \\ &= \sum_{\delta} \Pr(\Delta_i = \delta) \int_{t_i - .5}^{t_i + .5} \mathbf{1}(t + \delta; t_i - 0.5, \mathcal{A}_i) f_T(t)[1 - F_R(t)]dt \end{aligned} \quad (3)$$

where

$$\mathbf{1}(t + \delta; t_i - 0.5, \mathcal{A}_i) = \begin{cases} 1 & \text{when } t_i - 0.5 \leq t + \delta \leq \mathcal{A}_i \\ 0 & \text{otherwise} \end{cases}.$$

The indicator function  $\mathbf{1}(t + \delta; t_i - 0.5, \mathcal{A}_i)$  accounts for the censoring that arises because we only know about failures that are reported before the DFD. For purposes of numerical computation, equation (3) can be re-expressed as,

$$\pi_i = \int_{t_i - .5}^{t_i + .5} \sum_{\delta} \Pr(\Delta_i = \delta) \mathbf{1}(t + \delta; t_i - 0.5, \mathcal{A}_i) f_T(t)[1 - F_R(t)]dt.$$

For those units that were not reported as failures before DFD, the probability that a unit in installation batch  $j$  has not been reported as a failure before DFD is

$$\begin{aligned} \xi_j &= 1 - \Pr[(T \leq R) \cap (0 < T \leq \mathcal{A}_j) \cap \text{Reported}] \\ &= 1 - \sum_{\delta} \Pr(\Delta_j = \delta) \Pr[(T \leq R) \cap (0 < T \leq \mathcal{A}_j) \cap \text{Reported} | \delta] \\ &= 1 - \sum_{\delta} \Pr(\Delta_j = \delta) \int_0^{\mathcal{A}_j} \mathbf{1}(t + \delta; 0, \mathcal{A}_j) f_T(t)[1 - F_R(t)]dt \end{aligned} \quad (4)$$

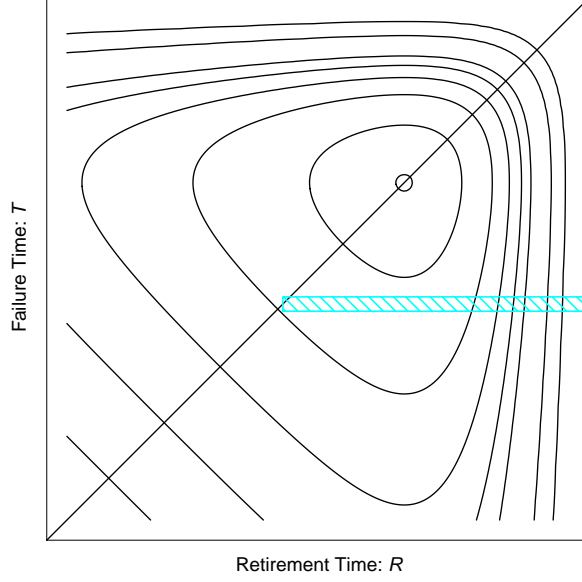


Figure 2. Illustration of the likelihood contribution in (2) relative to the joint distribution of  $T$  and  $R$ . The shaded area shows the likelihood contribution.

where

$$\mathbf{1}(t + \delta; 0, \mathcal{A}_j) = \begin{cases} 1 & \text{when } 0 \leq t + \delta \leq \mathcal{A}_j \\ 0 & \text{otherwise} \end{cases}.$$

Equation (4) can be re-expressed as

$$\xi_j = 1 - \int_0^{\mathcal{A}_j} \sum_{\delta} \Pr(\Delta_j = \delta) \mathbf{1}(t + \delta; 0, \mathcal{A}_j) f_T(t) [1 - F_R(t)] dt.$$

The log-likelihood function based on the data in Tables I and II is

$$\mathcal{L}(\boldsymbol{\theta} | \text{DATA}) = \sum_{i=1}^r \log(\pi_i) + \sum_{j=1}^J w_j \log(\xi_j), \quad (5)$$

where  $\boldsymbol{\theta} = (\eta_T, \beta_T)'$ . Here the first summation is over the reported failures, the second summation is over the installation batches in Table II and  $w_j$  is the number of units from batch  $j$  that have not been reported as failures.

### B. Parameter Estimates

The ML estimator of  $\boldsymbol{\theta}$  is denoted by  $\hat{\boldsymbol{\theta}} = (\hat{\eta}_T, \hat{\beta}_T)'$ . To make the numerical optimization more stable, we optimized the loglikelihood function using an alternative parametrization  $t_{0.001}$  and  $\beta_T$ , instead of the original parametrization  $\eta_T$  and  $\beta_T$ . Here  $t_{0.001}$  is the 0.001 quantile of the product failure-time distribution. The effect of the reparametrization is shown

Table IV  
ML ESTIMATES AND STANDARD ERRORS FOR PARAMETERS UNDER  $E(R) = 98, \beta_R = 1.5$ .

Parameter	ML est.	Std. err.	95% lower	95% upper
$\beta_T$	2.788	0.428	2.064	3.766
$\eta_T$	1670.902	730.451	709.316	3936.066

in Figure 3. Under the original parametrization, the shape of the log relative likelihood is elongated, indicating a strong correlation between  $\hat{\eta}_T$  and  $\hat{\beta}_T$ . Such strong correlation will make the numerical optimization less stable. Under the alternative parametrization, the log-likelihood is better behaved. Due to the invariance property of ML estimators, the ML estimates obtained under the alternative parametrization can be transformed to the ML estimates for the original parameters for subsequent computations.

Table IV shows the estimates of the Weibull shape and scale parameters based on the Weibull retirement distribution assumption with  $E(R) = 98$ , and  $\beta_R = 1.5$ . Under this assumption, the Weibull shape parameter estimate is  $\hat{\beta}_T = 2.788$ , which is larger than 1, indicating that the failure-time distribution hazard function is increasing, which is in agreement with the known physical degradation cause of failure.

It is important to consider retirement when one needs to determine the fraction reported. As an illustration, Figure 4 shows the failure-time distributions with and without adjustment of retirement, when  $E(R) = 98$  and  $\beta_R = 1.5$ . The figure illustrates the large effect that the retirement distribution plays in determining the fraction reported as a function of time. That is, the failure probability of the distribution without considering retirement is much larger than the one with retirement as time goes.

#### IV. PREDICTION OF FUTURE NUMBER OF REPORTS

##### A. Probability of Being Reported Before Retirement

Based on the model for the failure-time distribution and the ML estimates  $\hat{\eta}_T$  and  $\hat{\beta}_T$  and the assumed values of  $\eta_R$  and  $\beta_R$ , one can predict the number of reported failures that will occur before a specified future point in time. For any particular unit in batch  $j$  that has not been reported as a failure before the DFD may fail and be reported in the future time interval  $(\mathcal{A}_j + s - .5, \mathcal{A}_j + s + .5)$ , where  $s$  is the number of months after the DFD,  $s = 1, 2, 3, \dots$ .

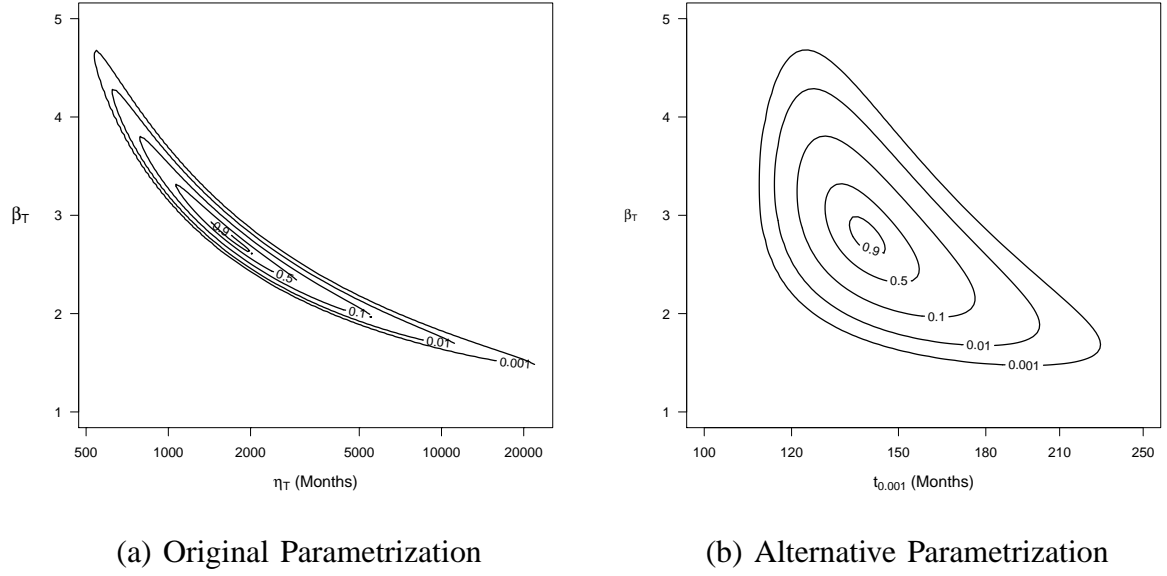


Figure 3. Comparison of shape of the log relative likelihood function of the two parametrizations with  $E(R) = 98$ , and  $\beta_R = 1.5$ .

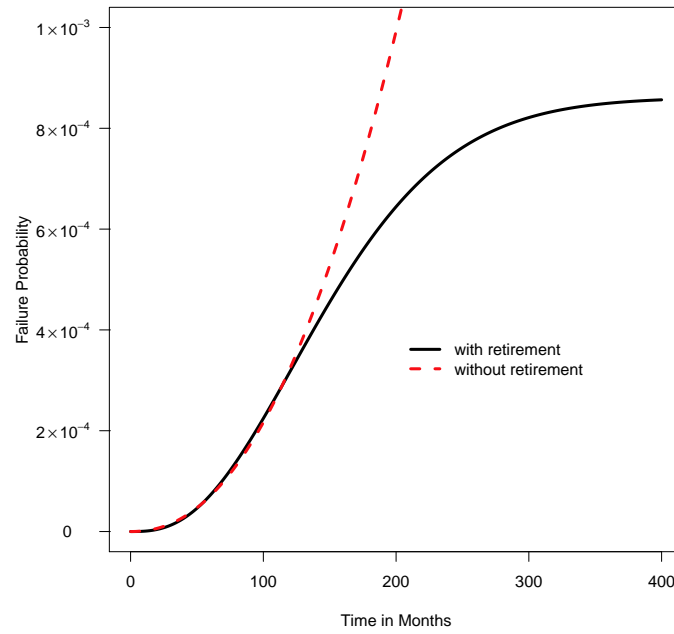


Figure 4. Failure-time distributions with and without adjustment of retirement when  $E(R) = 98$  and  $\beta_R = 1.5$ .

The corresponding probability is

$$\begin{aligned}
 h_j(s) &= \Pr(T + \Delta \in [\mathcal{A}_j + s - .5, \mathcal{A}_j + s + .5], T \leq R \mid \text{not being reported by } \mathcal{A}_j) \\
 &= \frac{\Pr(T + \Delta \in [\mathcal{A}_j + s - .5, \mathcal{A}_j + s + .5], T \leq R, \text{not being reported by } \mathcal{A}_j)}{\Pr(\text{not being reported by } \mathcal{A}_j)} \\
 &= \frac{\Pr(T + \Delta \in [\mathcal{A}_j + s - .5, \mathcal{A}_j + s + .5], T \leq R)}{1 - \Pr(\text{being reported by } \mathcal{A}_j)}. \tag{6}
 \end{aligned}$$

Here,  $T$  denotes the failure time of a unit that has not been reported as failures by the DFD,  $\Delta$  denotes the random reporting delayed time,  $j$  denotes the batch number,  $j = 1, \dots, J$ , and  $\mathcal{A}_j$  denotes the age of the units in batch  $j$  at the DFD. Let

$$\gamma_j(s) = \Pr(T + \Delta \in [\mathcal{A}_j + s - .5, \mathcal{A}_j + s + .5], T \leq R),$$

and let  $\xi_j = 1 - \Pr(\text{being reported by } \mathcal{A}_j)$  in (6). In particular,

$$\begin{aligned}
 \gamma_j(s) &= \Pr[T + \Delta \in [\mathcal{A}_j + s - .5, \mathcal{A}_j + s + .5], T \leq R] \\
 &= \sum_{\delta} \int_0^{\mathcal{A}_j + s + .5} \Pr[T + \Delta \in [\mathcal{A}_j + s - .5, \mathcal{A}_j + s + .5], T \leq R \mid T = t, \Delta = \delta] f_T(t) \Pr(\Delta = \delta) dt \\
 &= \sum_{\delta} \int_0^{\mathcal{A}_j + s + .5} \mathbf{1}(t + \delta; \mathcal{A}_j + s - .5, \mathcal{A}_j + s + .5) [1 - F_R(t)] f_T(t) \Pr(\Delta = \delta) dt \\
 &= \sum_{\delta} \Pr(\Delta = \delta) \int_0^{\mathcal{A}_j + s + .5} \mathbf{1}(t + \delta; \mathcal{A}_j + s - .5, \mathcal{A}_j + s + .5) (1 - F_R(t)) f_T(t) dt
 \end{aligned}$$

where

$$\mathbf{1}(t + \delta; \mathcal{A}_j + s - 0.5, \mathcal{A}_j + s + 0.5) = \begin{cases} 1 & \text{when } \mathcal{A}_j + s - 0.5 \leq t + \delta \leq \mathcal{A}_j + s + 0.5 \\ 0 & \text{otherwise} \end{cases}.$$

Here the indicator function  $\mathbf{1}(t + \delta; \mathcal{A}_j + s - .5, \mathcal{A}_j + s + .5)$  constrains the reporting time between  $\mathcal{A}_j + s - .5$  and  $\mathcal{A}_j + s + .5$ . The factor  $1 - F_R(t)$  represents the probability that the unit has not retired before it fails. Numerical integration is needed to compute  $\gamma_j$ . The quantity  $\xi_j$  is computed as follows,

$$\begin{aligned}
 \xi_j &= 1 - \Pr[\text{being reported by } \mathcal{A}_j] \\
 &= 1 - \Pr[(T \leq R) \cap (0 < T \leq \mathcal{A}_j) \cap \text{Reported}] \\
 &= 1 - \sum_{\delta} \Pr(\Delta_j = \delta) \Pr[(T \leq R) \cap (0 < T \leq \mathcal{A}_j) \cap \text{Reported} \mid \delta] \\
 &= 1 - \sum_{\delta} \Pr(\Delta_j = \delta) \int_0^{\mathcal{A}_j} \mathbf{1}(t + \delta; 0, \mathcal{A}_j) f_T(t) [1 - F_R(t)] dt \tag{7}
 \end{aligned}$$



which is similar to (4). The only difference is that we use (4) to estimate parameters, but use (7) evaluated at the ML estimates to provide an estimate of the probability that a unit in batch  $j$  is not reported as a failure before  $\mathcal{A}_j$ . Based on the probability function,  $h_j(s) = \gamma_j(s)/\xi_j$ , the probability that a unit in batch  $j$  is reported as a failure before time point  $s$  is

$$\rho_j(s) = \sum_{l=1}^s h_j(l), \quad s = 1, 2, \dots \quad (8)$$

Because the ages of units at the DFD from different batches are not the same, the  $\rho_j(s)$ 's are different for different batches.

### B. Point Prediction

For unit  $k$  in batch  $j$  that has not been reported as a failure by  $t_j^U$ , we use  $I_{jk}(s)$  as an indicator for being reported in future time interval  $(0, s]$ . The distribution of  $I_{jk}(s)$  is Bernoulli $[\rho_j(s)]$ . Thus the cumulative number of reported failures at time  $s$  is

$$N(s) = \sum_{j=1}^J \sum_{k=1}^{w_j} I_{jk}(s), \quad (9)$$

which is the sum of independent and non-identical indicators. The point prediction (estimate of the expected number failing) for the number of reports up to time  $s$  is

$$\hat{N} = \hat{N}(s) = \sum_{j=1}^J w_j \hat{\rho}_j(s).$$

Here,  $w_j$  is the number of units not reported as having failed by the DFD in batch  $j$ , and  $\hat{\rho}_j(s)$  is obtained by evaluating (8) at the ML estimates  $\eta_T$  and  $\beta_T$  and the assumed values of  $\eta_R$  and  $\beta_R$ .

### C. Prediction Interval

This section introduces a method of computing PIs for the cumulative number of reported failures at a future time point. The cumulative number of reported failures  $N(s)$ , as given in (9), is the sum of independent and non-identically distributed Bernoulli random variables which follows a Poisson-binomial distribution. Hong [15] gives an exact expression for the cdf of the Poisson-binomial distribution based on a discrete Fourier transform. In particular, the cdf of  $N(s)$ , denoted by  $F_N(n)$ ,  $n = 0, 1, \dots, n^*$ , is

$$F_N(n) = \frac{1}{n^* + 1} \sum_{l=0}^{n^*} \left\{ \frac{\exp(-i\omega l n) - \exp(-i\omega l)}{1 - \exp(-i\omega l)} \prod_{j=1}^J [1 - \rho_j(s) + \rho_j(s) \exp(i\omega l)]^{w_j} \right\}$$

where  $n^* = \sum_j w_j$ ,  $\mathbf{i} = \sqrt{-1}$  and  $\omega = 2\pi/(n^* + 1)$ .

Using the predictive distribution given in Lawless and Fredette [12], a  $100(1 - \alpha)\%$  PI for  $N = N(s)$ , denoted by  $[\underline{N}, \tilde{N}]$ , is obtained by solving

$$F_N(\underline{N}; \hat{\boldsymbol{\theta}}) = v_{\alpha/2} \quad \text{and} \quad F_N(\tilde{N}; \hat{\boldsymbol{\theta}}) = v_{1-\alpha/2}. \quad (10)$$

Here  $v_\alpha$  is the  $\alpha$  quantile of the distribution of the random quantity  $F_N(N; \hat{\boldsymbol{\theta}})$ , where both  $N$  and  $\hat{\boldsymbol{\theta}}$  are treated as random variables. We use a bootstrap simulation procedure to approximate the quantile  $v_\alpha$ . In particular,  $v_\alpha$  is approximated from  $B$  bootstrap samples by the  $\alpha$  sample quantile of  $F_N(N^{*b}; \hat{\boldsymbol{\theta}}^{*b})$ ,  $b = 1, \dots, B$ . Here  $N^*$  is simulated from  $F_N(n; \hat{\boldsymbol{\theta}})$  given the ML estimate  $\hat{\boldsymbol{\theta}}$ , and  $\hat{\boldsymbol{\theta}}^*$  is the ML estimates obtained from bootstrap samples. We use the random weighted bootstrap proposed by Newton and Raftery [16] instead of the ordinary bootstrap because of the heavy censoring and the complicated data structure. The specific procedure for such a bootstrap is described as follows:

- 1) Simulate random weights  $W_i$  and  $W_{jk}$  from a positive distribution with the property of  $\text{Var}(W) = [\text{E}(W)]^2$ , where  $W_i$  is the random weight for reported failure unit  $i$  and  $W_{jk}$  is the random weight the not-reported unit  $k$  in batch  $j$  by the DFD. We sample  $W_i$  and  $W_{jk}$  from the exponential distribution with mean of one.
- 2) Based on the random weights  $W_i$  and  $W_{jk}$ , we calculate the random weighted likelihood,

$$\mathcal{L}^*(\boldsymbol{\theta} | \text{DATA}) = \sum_{i=1}^r W_i \log(\pi_k) + \sum_{j=1}^J \sum_{k=1}^{w_j} W_{jk} \log(\xi_j). \quad (11)$$

- 3) Obtain the estimated  $\hat{\boldsymbol{\theta}}^*$  by maximizing (11).
- 4) Repeat the above steps  $B$  times to obtain the bootstrap samples  $\hat{\boldsymbol{\theta}}^{*b}$ ,  $b = 1, 2, \dots, B$ .

Following Lawless and Fredette [12], we construct a calibrated PI for  $N$  by using the following steps:

- 1) Simulate  $N^{*b} \sim F_N(n, \hat{\boldsymbol{\theta}})$ ,  $b = 1, 2, \dots, B$ , where  $\hat{\boldsymbol{\theta}}$  is the vector of ML estimates from the original data.
- 2) Compute  $v^b = F_N(N^{*b}, \hat{\boldsymbol{\theta}}^{*b})$ ,  $b = 1, 2, \dots, B$ .
- 3) Calculate the lower and upper  $\alpha/2$  quantiles of  $\{v^b, b = 1, 2, \dots, B\}$ , denoted by  $v_{\alpha/2}$  and  $v_{1-\alpha/2}$ , respectively.
- 4) Solve  $\underline{N}$  from  $F_N(\underline{N}, \hat{\boldsymbol{\theta}}) = v_{\alpha/2}$  and  $\tilde{N}$  from  $F_N(\tilde{N}, \hat{\boldsymbol{\theta}}) = v_{1-\alpha/2}$  to obtain the endpoints of the PI.

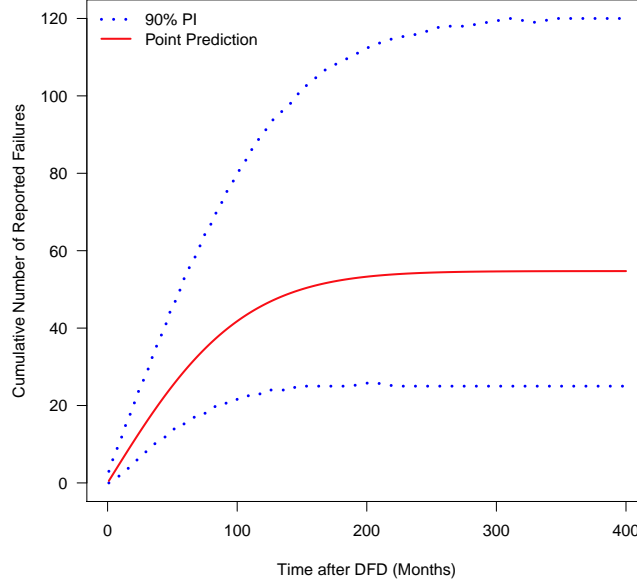


Figure 5. Plot of the 90% PI for cumulative number of reported failures based on Weibull distribution assumption for retirement and failure distributions, with  $E(R) = 98$  and  $\beta_R = 1.5$ .

#### D. Prediction Results

In this section, we present the results of point predictions and PIs for the cumulative number of reported failures, based on the assumptions that both of the failure-time distribution and the retirement-time distribution are Weibull. Figure 5 is plotted based on  $E(R) = 98$  and  $\beta_R = 1.5$ . It shows that the cumulative number of reported failures is increasing rapidly until 150 months, and is approximately constant after 250 months, indicating that the reported failures will be rare after and additional 250 months (approximately 11 years). The leveling-off is caused by the fact that most units retire before 250 months. For the 90% PI, the upper bound is around 120 and the lower bound is around 25 after 200 months. Compared to the initial number of units not reported as failures ( $w = 120,889$ ), the predicted cumulative number of reported failures (estimate of the expected number) is around 55, which is a small amount of the units, relative to the number that had been put into service (i.e., less than 0.046%).

### V. SENSITIVITY ANALYSIS

The prediction of the cumulative number of reported failures is based on uncertain assumptions including the failure-time distribution as well as the parameters and distribution

Table V  
ML ESTIMATES AND STANDARD ERRORS FOR MODEL PARAMETERS.

Assumed Retirement-time			Failure-time Distribution				
Distribution Parameters			Estimates				
$E(R)$	$\eta_R$	$\beta_R$	$\hat{\eta}_T$	$\hat{\beta}_T$	$se(\hat{\eta}_T)$	$se(\hat{\beta}_T)$	−Log likelihood
85	94.157	1.5	1390.523	2.928	555.691	0.436	436.927
90	99.696	1.5	1501.248	2.868	623.215	0.432	436.976
98	108.558	1.5	1670.901	2.788	730.451	0.428	437.047
85	95.912	2	1340.798	2.995	533.781	0.449	436.736
90	101.554	2	1486.736	2.908	622.556	0.443	436.805
98	110.581	2	1712.534	2.796	766.316	0.435	436.908

for retirement times. Changes in these assumptions will affect the prediction results. Thus, it is necessary to do the sensitivity analysis to assess the effect of departures from the assumptions and to understand which assumptions are conservative.

#### A. Parameter Assumptions

The prediction results shown in Figure 5 are based on the Weibull distribution for the retirement model with  $E(R) = 98$  and  $\beta_R = 1.5$ . Historical information suggests that the values of parameters in the retirement-time distribution are within a certain range. Thus, it is desirable to consider other retirement-time model parameters to assess the effect that deviations from the assumptions have on the prediction results. Table V and Figure 6 summarize the results of this sensitivity analysis. From Table V, we note that, as expected, the ML estimates  $\hat{\eta}_T$  and  $\hat{\beta}_T$  change under different assumptions for the retirement distribution parameters. There is more change in the estimates of  $\hat{\eta}_T$  because these estimates involve a substantial amount of extrapolation. The maximum log-likelihood values are close to each other indicating there is little or no information about the retirement distribution parameters in the data. Figure 6 shows corresponding predictions for the cumulative number of reported failures. The graph indicates that the predicted cumulative number of reported failures increases as the expected retirement time and the Weibull shape parameter increase. Compared to the change of expected retirement time, predictions are more sensitive to the assumption about the Weibull shape parameter. But relative to the large number units that had been put into service, the differences in the predictions for the number of future reported failures is not large.

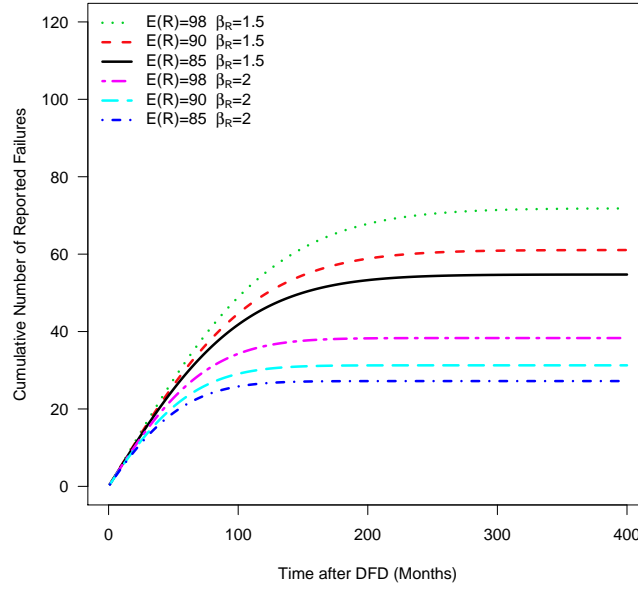


Figure 6. Weibull distribution point predictions for the cumulative number of reported failures with different values for the parameters in the retirement distributions.

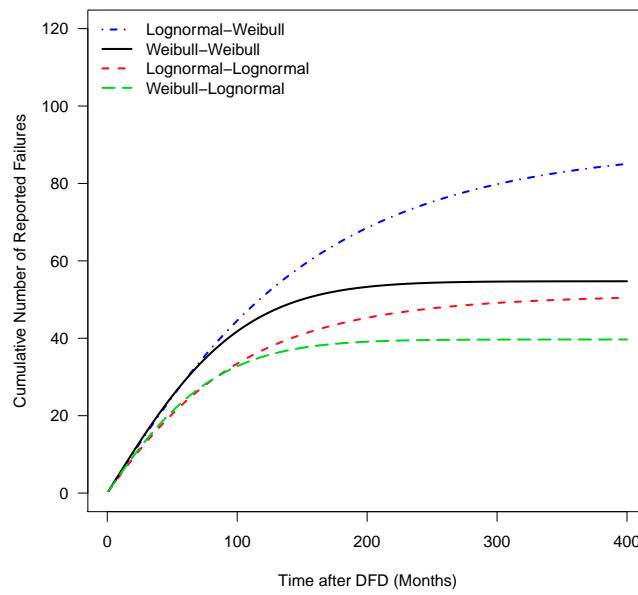


Figure 7. The predicted number of the reported failures based on different failure-time and retirement-time distributions when  $E(R) = 85$  and  $SD(R) = 57.7$ . The legend shows the failure-time and retirement-time distribution combinations.

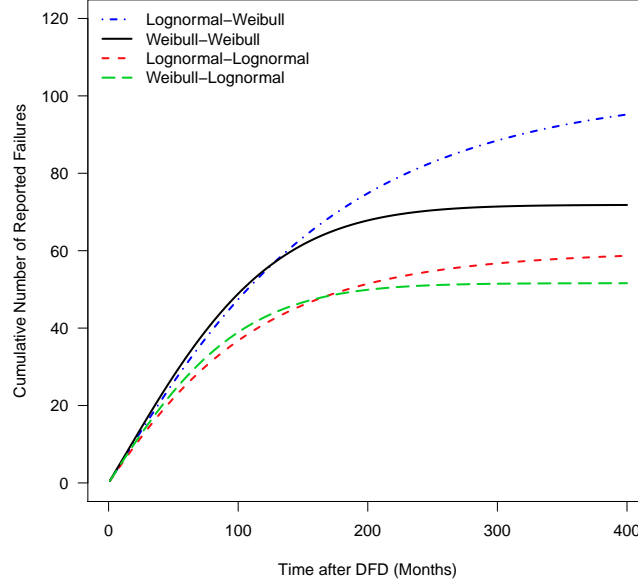


Figure 8. The predicted number of the reported failures based on different failure-time and retirement-time distributions when  $E(R) = 98$  and  $SD(R) = 66.5$ . The legend shows the failure-time and retirement-time distribution combinations.

### B. Distributional Assumptions

In the previous analysis, the retirement and the failure-time distributions were assumed to be Weibull. The data, however, do not provide much information to distinguish among competing distributions. Thus, it is useful compute predictions with different retirement-time distribution and failure-time distribution assumptions. In this sensitivity analysis, we use the lognormal distribution as an alternative for the failure-time distribution and the retirement-time distribution. When the lognormal distribution is chosen as the retirement-time distribution, the assumed mean and standard deviation are specified to be the same as that assumed for the Weibull retirement-time distribution.

Table VI shows the ML estimates for the failure-time distribution parameters when  $E(R) = 85$  and  $SD(R) = 57.7$  for the four combinations of the failure-time and retirement distributions. From the results in Table VI, the maximum log-likelihood values are quite close for all four combinations. When the failure-time distribution is the same, there is no obvious difference in the value of  $\hat{\mu}_T$  and  $\hat{\sigma}_T$ . Table VII shows similar results when  $E(R) = 98$  and  $SD(R) = 66.5$ .

Figure 7 shows the predicted cumulative number of reported failures with different retirement-time and failure-time distributions. The legend shows the failure-time and retirement-time

Table VI  
ML ESTIMATES FOR PARAMETERS WHEN  $E(R) = 85, SD(R) = 57.7$ .

Assumed Retirement-time Distribution	Assumed Failure-time Distribution	$\hat{\mu}_T$	$\hat{\sigma}_T$	−Log likelihood
Weibull	Weibull	7.237	0.341	436.927
Weibull	Lognormal	8.893	1.292	436.640
Lognormal	Weibull	7.044	0.319	436.801
Lognormal	Lognormal	8.594	1.210	436.572

Table VII  
ML ESTIMATES FOR PARAMETERS WHEN  $E(R) = 98, SD(R) = 66.5$ .

Assumed Retirement-time Distribution	Assumed Failure-time Distribution	$\hat{\mu}_T$	$\hat{\sigma}_T$	−Log likelihood
Weibull	Weibull	7.421	0.359	437.047
Weibull	Lognormal	9.185	1.364	436.735
Lognormal	Weibull	7.255	0.339	436.878
Lognormal	Lognormal	8.929	1.291	436.626

distribution combinations. For example, “Lognormal-Weibull” indicates that retirement times follow a lognormal distribution and failure times follow a Weibull distribution. Figure 8 shows similar comparisons under a different set of values for  $E(R)$  and  $SD(R)$ . The effect of different retirement-time distributions is much stronger when the failure-time distribution is Weibull. Compared to other distribution combinations, a lognormal retirement distribution with a Weibull failure-time distribution provides the most conservative predictions (i.e., predicts more reported failures).

Compared with the large number of product units in the field (there were 120,921 units put into service), the differences of predicted cumulative number of reported failures among the four distribution combinations are small. Thus, the predictions of reported number of future failures do not depend strongly on the assumed retirement-time and failure-time distributions.

## VI. CONCLUSIONS AND AREAS FOR FUTURE RESEARCH

This paper provides general statistical methods to predict the field failures and conduct a risk assessment for products. To generate accurate predictions, the proposed method considers the effect of retirement times and reporting delays when estimating the failure-time distribution and when making predictions. Based on the failure-time model, we predict the cumulative number of reported failures and construct a corresponding PI. We also conduct sensitivity analysis to assess the effect of different failure-time and retirement-time distributions.

There are some possible areas for future research.

- In the product B application, the degradation process that causes the serious failures was related to product age (not the amount of use) and was unrelated to customer-perceivable performance degradation. Thus it was reasonable to assume that the retirement-time and failure-time random variables were independent. In other applications, a model that allows dependency could be used.
- For most products, it is not possible to track the retirement time for all units. For some applications it would be possible and useful to track a representative subset of the product populations through a carefully designed field tracking study.
- Today, some products, even home appliances, can be connected to the internet, potentially providing detailed information about how each such unit is being used. See Hong and Meeker [17], and Hong and Meeker [18] for applications involving prediction of future failures when a proportion of units in the product population are connected to the Internet. Having the additional information about which units are still in active use would reduce much of the uncertainty in predictions associated with a risk analysis.
- For the product B application, there was only limited information about the retirement-time distribution, based on a completely separate marketing study for a similar product. An alternative analysis could have taken that information, perhaps supplemented by expert opinion, to develop a joint prior distribution to describe unknown characteristics (including the form and the parameters) of the retirement-time and failure-time distributions. This would allow a fully Bayesian analysis to be performed.

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